

Metabolic Network Analysis

Christoph Flamm

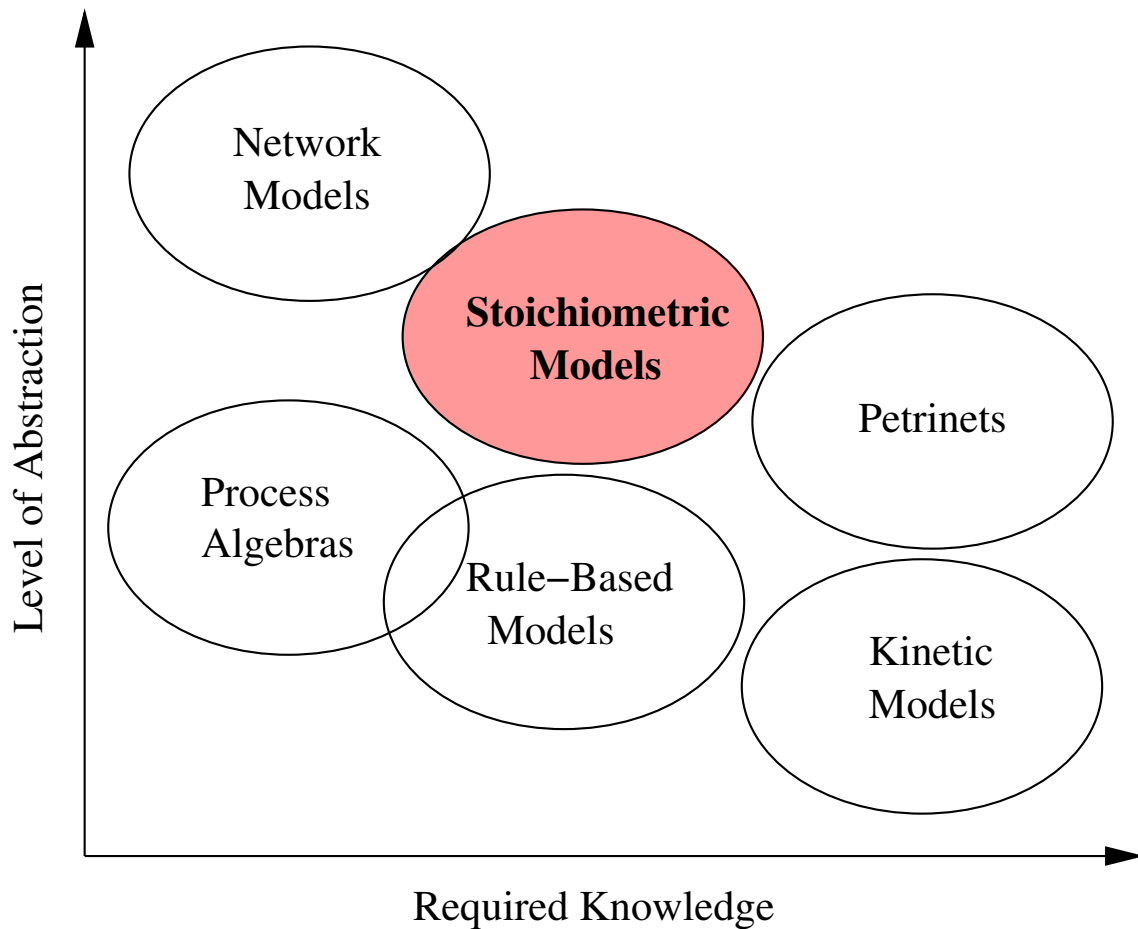
xtof@tbi.univie.ac.at

Institute for Theoretical Chemistry
University of Vienna

VS 791328 Modern Bioinformatics
Wien, December 14, 2016



The Systems Biology Modelling Space



Genome-scale Reconstruction of Metabolic Networks

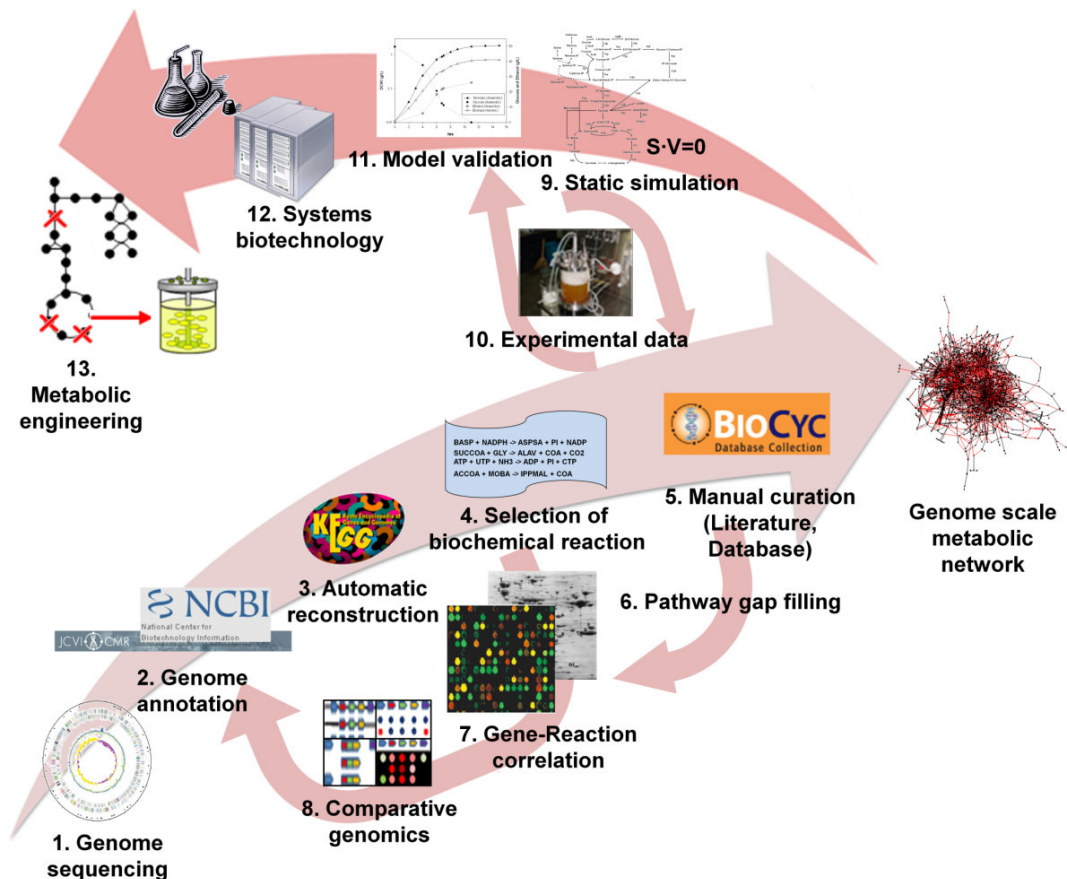
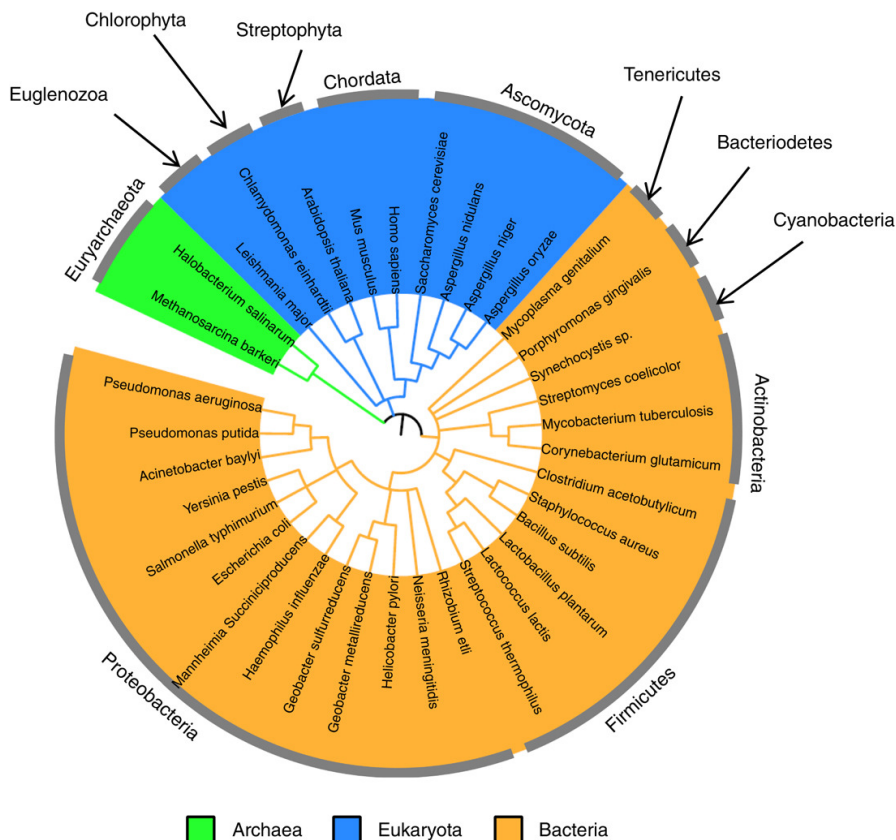


Figure from Lee KY et al (2010), Microbial Cell Factories 9:94 | doi:10.1186/1475-2859-9-94

Phylogenetic tree of metabolic reconstructions

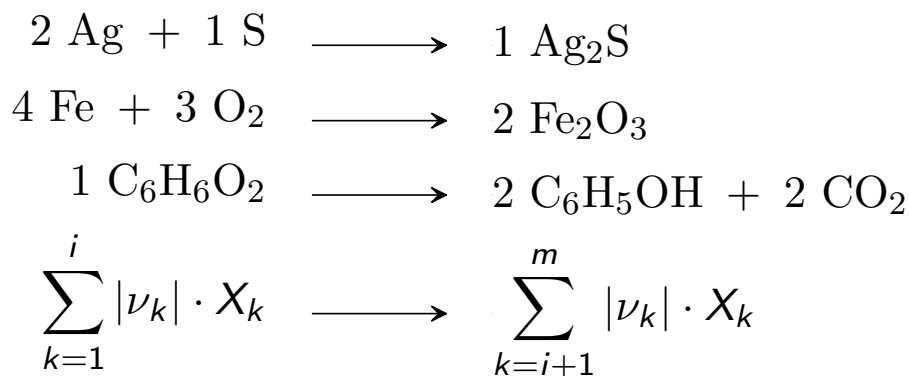


Our metabolic knowledge is strongly biased towards cultivatable bacteria.

Figure from Oberhardt MA et al (2009) Mol Syst Biol 5:320 | doi:10.1038/msb.2009.77

Chemical Reactions and Stoichiometry

Educt(s) are found left and **product(s)** right of the reaction arrow.



The ν_k 's are *integer values* called **stoichiometric coefficients**.

By convention:

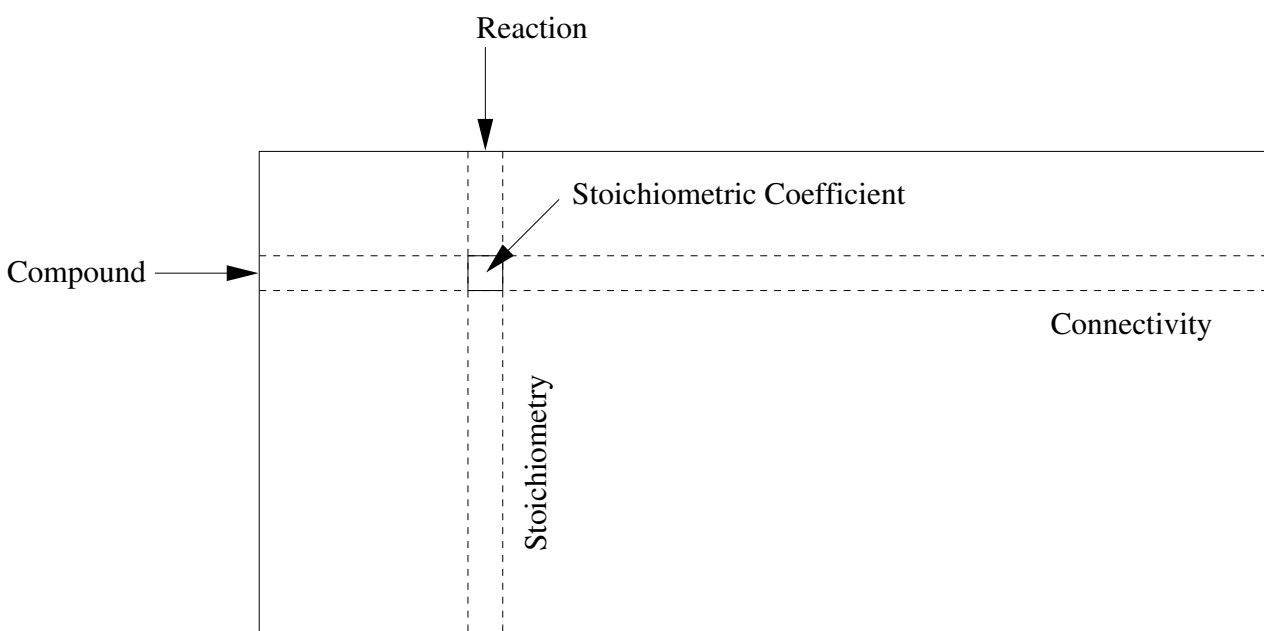
$$\nu_k = \begin{cases} > 0 & \text{if compound is formed by reaction} \\ < 0 & \text{if compound is consumed by reaction} \\ = 0 & \text{otherwise} \end{cases}$$

Note that the stoichiometric coefficients are **constants** and do not depend on reaction conditions such as temperature, pressure, pH, . . .

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Stoichiometric Matrix S

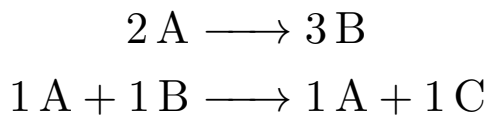
It is convenient to handle sets of chemical reactions, composing a reaction network, in **matrix form**.



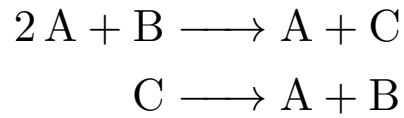
The **rows** correspond to *chemical compounds*, and **columns** to **reactions**.

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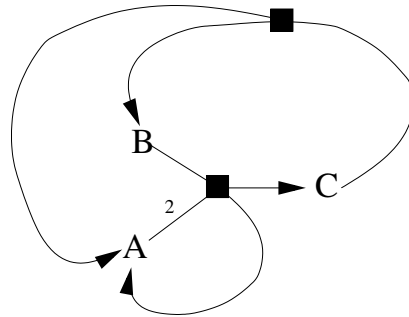
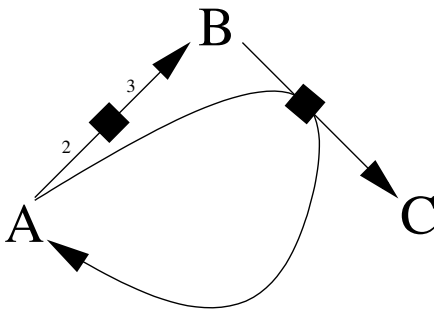
Examples: Stoichiometric Matrix



$$S = \begin{pmatrix} -2 & 0 \\ 3 & -1 \\ 0 & 1 \end{pmatrix}$$



$$S = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$



The three representations are equivalent apart from catalysts.

Genome-scale Stoichiometric Matrix

are sparse objects.

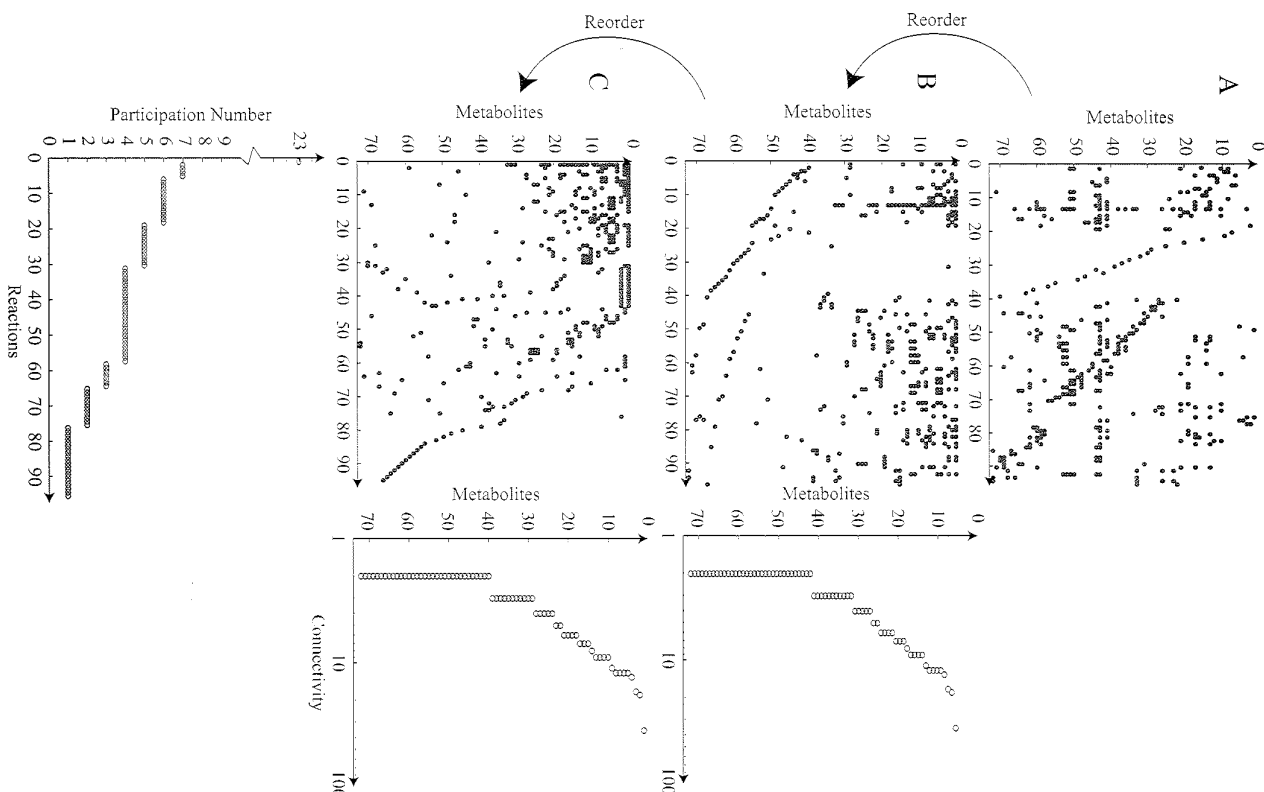
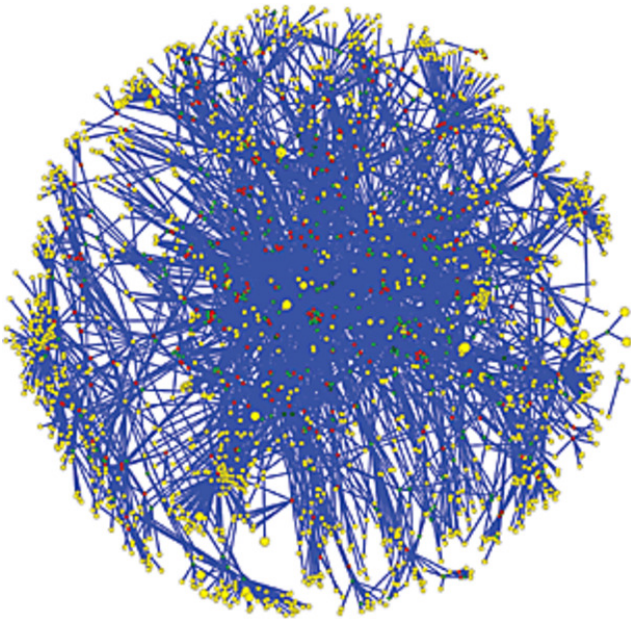


Figure modified from Palsson, BØ (2015), Systems Biology: Constraint-based Reconstruction and Analysis, ISBN 978-1-107-03885-1

What is a Biological Pathway?



$$S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix}$$

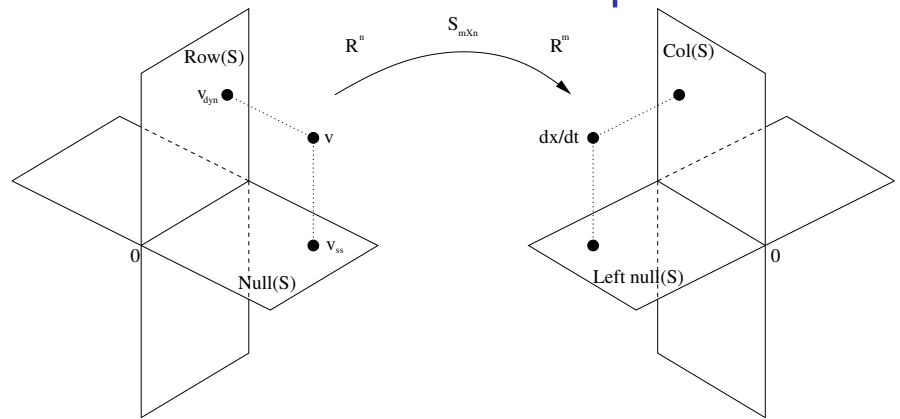
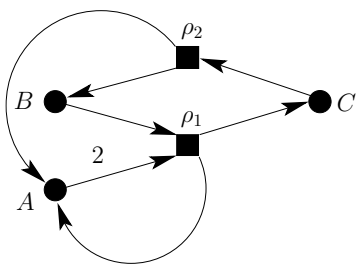
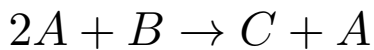
- ① Mathematical description of what a pathway is.
- ② Formalism to describe mass flows in the network.

Wanted: All **minimal sets of reaction**, which *can operate* at steady state and *respect* reaction directions.

Such a set is call an elementary pathway.

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The stoichiometric Matrix is a linear Operator



Reaction velocity Domain

Concentration Domain

$$\frac{d}{dt}[\vec{X}] = S \cdot \vec{J} = \frac{d}{dt} \begin{pmatrix} [A] \\ [B] \\ [C] \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} k_{\rho_1} [A]^2 [B] \\ k_{\rho_2} [C] \end{pmatrix}$$

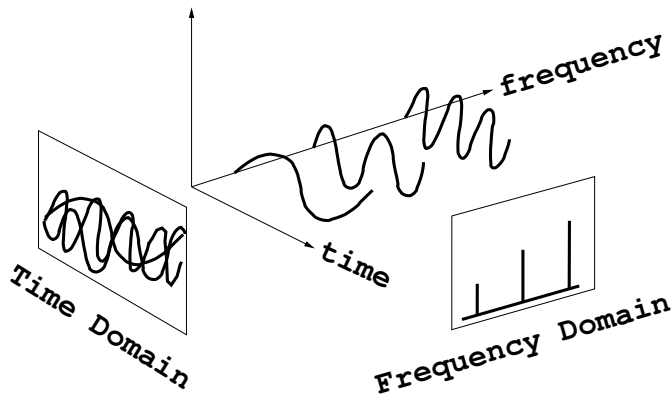
$$[\dot{A}] = k_{\rho_2} \cdot [C] - k_{\rho_1} \cdot [A]^2 \cdot [B]$$

$$[\dot{B}] = k_{\rho_2} \cdot [C] - k_{\rho_1} \cdot [A]^2 \cdot [B]$$

$$[\dot{C}] = k_{\rho_1} \cdot [A]^2 \cdot [B] - k_{\rho_2} \cdot [C]$$

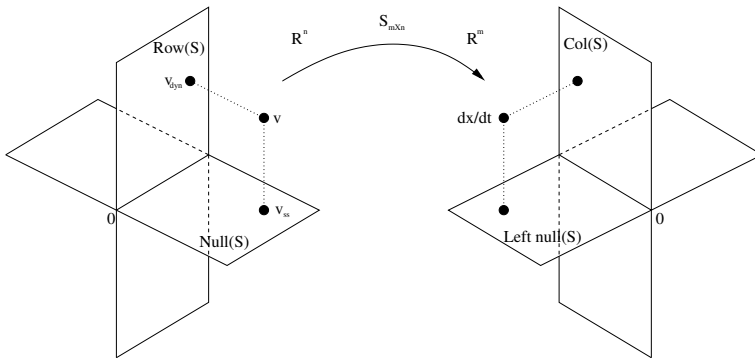
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Linear transformations between (vector) spaces



$$\frac{d}{dt}\vec{x} = \mathbf{S} \cdot \vec{v}$$

S as linear transformation from \vec{v} to $d\vec{x}/dt$.

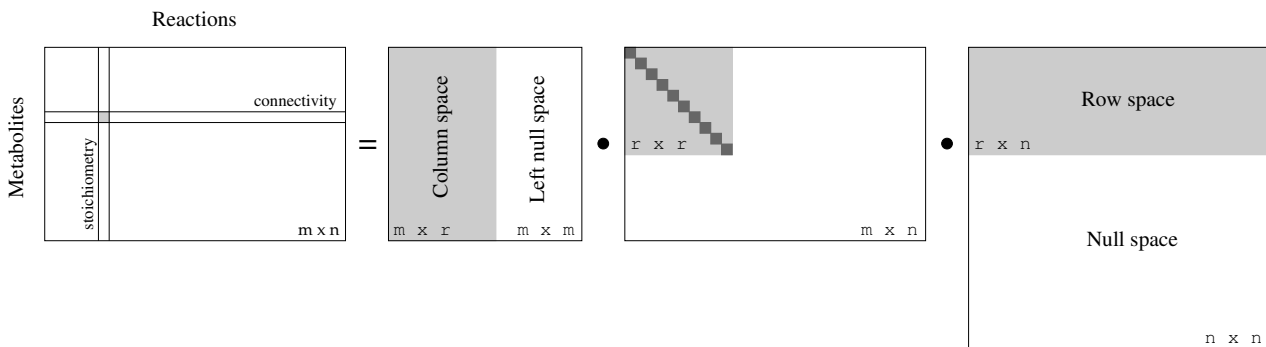


- **Flux domain**
Row space
Null space.
- **Conc. domain**
Column space
Left null space.

A probably familiar linear operator is the Fourier transform switching between time and frequency domain of a signal.

Singular Value Decomposition (SVD)

$$\mathbf{S} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^T$$

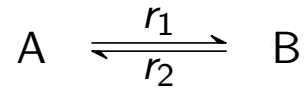


$$\mathbf{S} = \sum_{i=1}^r \sigma_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i^T)$$

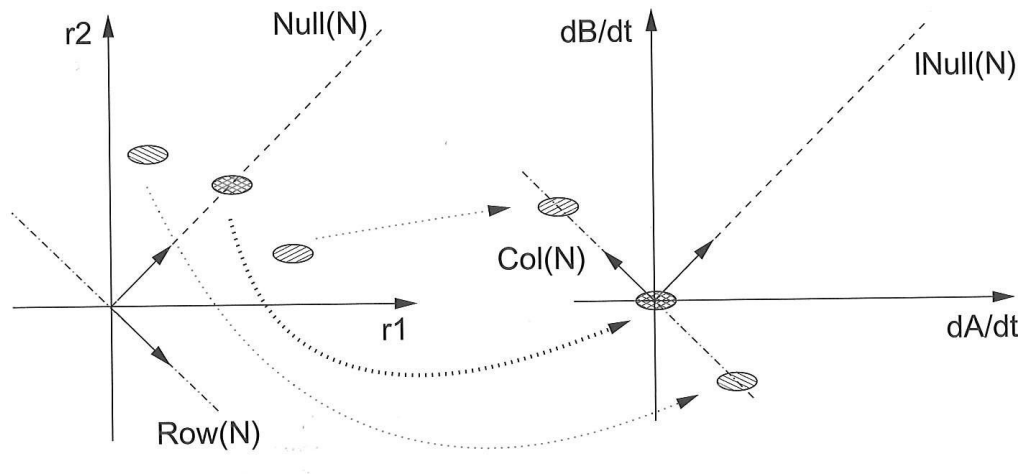
- 1 Row space contains all dynamic flux distributions.
- 2 Null space contains all steady-state flux distributions.
- 3 Column space contains all time derivatives of the conc. vector.
- 4 Left null space contains all conservation relationships.

r is the rank of S , i.e. the number of linear-independent row or column vectors

Example: SVD of a Reversible Reaction



$$N = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_U \cdot \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}}_\Sigma \cdot \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{V^T} \cdot \frac{1}{\sqrt{2}}$$



Relation between the 4 fundamental sub-spaces (Row(N), Null(N), Col(N), INull(N)) of the stoichiometric matrix N with rank $r = 1$. Figure adapted from Palsson *BØ Systems Biology – Properties of reconstructed networks*.

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The Kernel Matrix \mathbf{K} is a Null Space Basis

In steady-state the differential change in species concentrations vanishes

$$\frac{d}{dt}[\vec{X}] = \vec{0} = \mathbf{S} \cdot \vec{J}$$

Non-trivial solutions for the flux vector \vec{J} exists only if there are **linear dependencies** between columns of \mathbf{S}

$$\text{Rang}(\mathbf{S}) < \text{number of reactions}$$

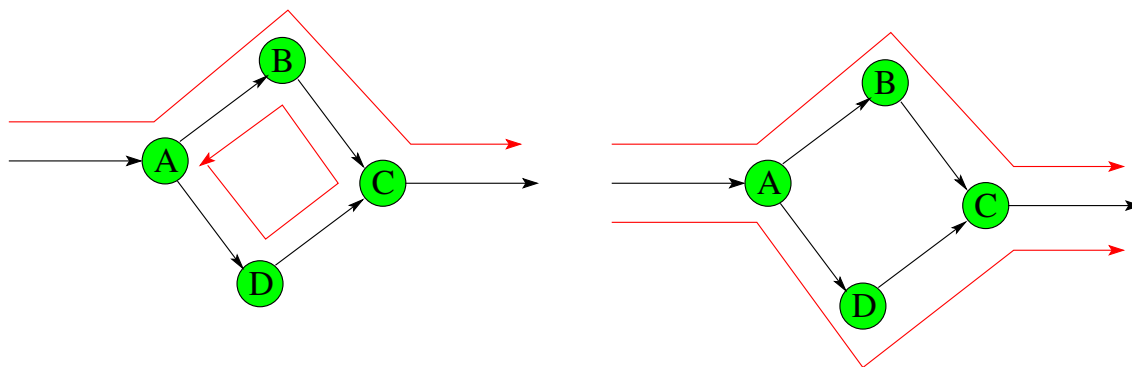
These **vectors** which **span the null space** of \mathbf{S} are most conveniently organized in a kernel matrix \mathbf{K} .

- Each column vector \vec{k}_i of \mathbf{K} solves $\mathbf{S} \cdot \vec{k} = 0$
- Any **admissible flux** in steady-state can be written as a **linear combination** of vectors \vec{k}_i

$$\vec{J} = \sum_i \alpha_i \cdot \vec{k}_i$$

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Orthonormal Null Space Basis cannot be Interpreted!

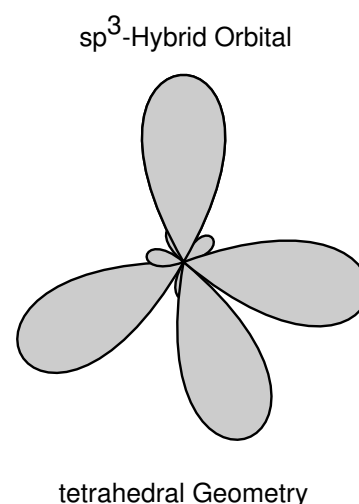
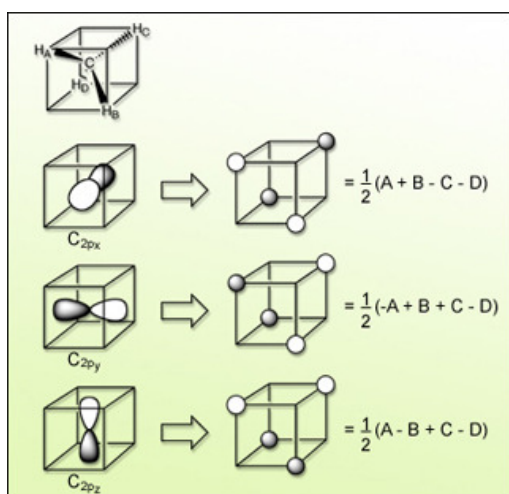


$$\mathbf{S} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{K}' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

- \mathbf{K} is not interpretable in chemically meaningful terms.
- Flux through elementary reactions must be positive.
- Non-negative basis vectors are required.
- A **convex basis** has proven useful for this goal.

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Hybrid orbitals: are a basis change!

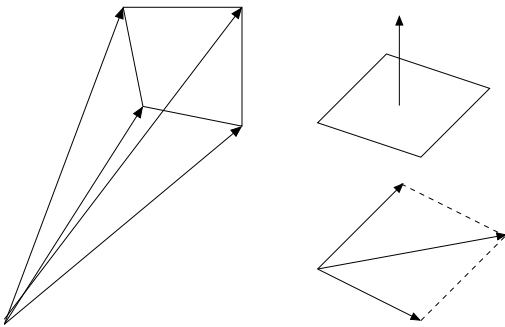


Linear combinations (hybrid orbitals) of H 1s atomic orbitals that match nodal properties of C 2p atomic orbitals to understand the tetrahedral geometry of methane.

Double Description Method for Basis Change

A pair (\mathbf{A}, \mathbf{R}) of real matrices \mathbf{A} and \mathbf{R} is said to be a double description pair if the following relationship holds

$$\mathbf{A}\mathbf{x} \geq \mathbf{0} \quad \text{if and only if} \quad \mathbf{x} = \mathbf{R}\boldsymbol{\lambda} \quad \text{for some } \boldsymbol{\lambda} \geq \mathbf{0}$$



\mathbf{A} is the *representation matrix*, (implicit description by constraints) ,
 \mathbf{R} is the *generating matrix* (explicit description by the edges) of a *polyhedral cone* P .

Geometrically, the columns of a minimal generating matrix \mathbf{R} are in a 1-to-1 correspondence with the *extreme rays* of P .

An iterative procedure initializing \mathbf{R} with \mathbf{K} is used to transform the *ortho-normal* to a *convex basis* respecting reaction directions.

Gagneur J and Klamt S, (2004) Computation of elementary modes: a unifying framework and the new binary approach, *BMC Bioinformatics* 5:175 | doi:10.1186/1471-2105-5-175

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Properties of linear and convex bases

Linear Space

- $\mathbf{A} \cdot \vec{x} = \vec{0}$.
- Set of linearly independent basis vectors (\vec{e}_i) .

$$\vec{v} = \sum w_i \vec{e}_i$$

with $w_i \in [-\infty, +\infty]$

- Unique representation of every point.
- $|\vec{e}_i| = \dim(\text{null}(\mathbf{S}))$.
- Infinit number of spanning bases.

Convex Space

- $\mathbf{A} \cdot \vec{x} = \vec{0}$ with $\vec{x} \geq \vec{0}$.
- Set of conically independent generating vectors (\vec{p}_i) .

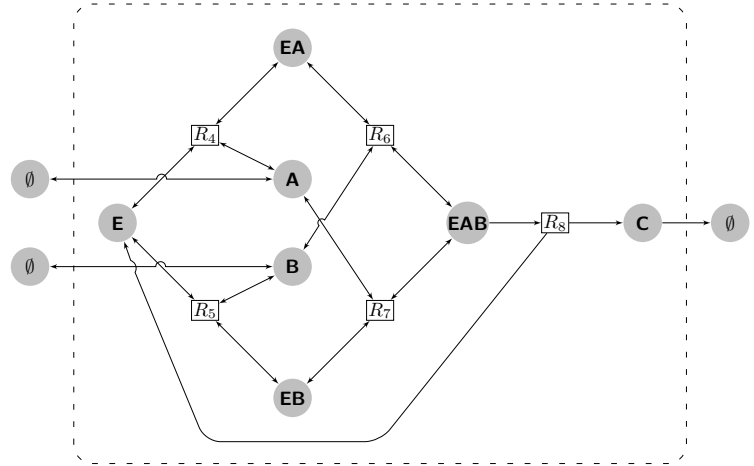
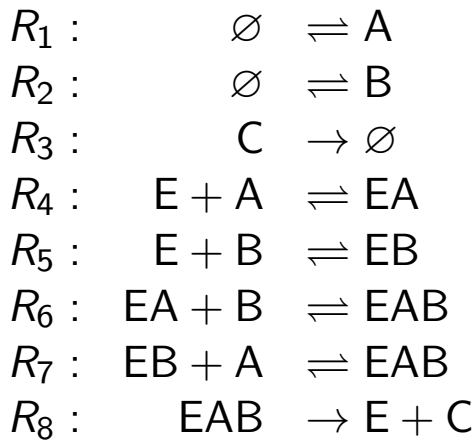
$$\vec{v} = \sum \alpha_i \vec{p}_i$$

with $\alpha_i \in [0, +\text{inf}]$

- Nonunique representation of every point.
- $|\vec{p}_i| \geq \dim(\text{null}(\mathbf{S}))$.
- Unique set of generating vectors.

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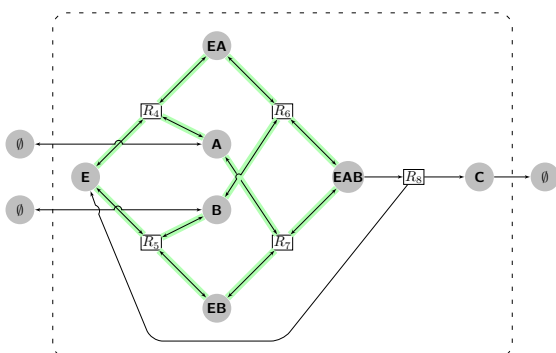
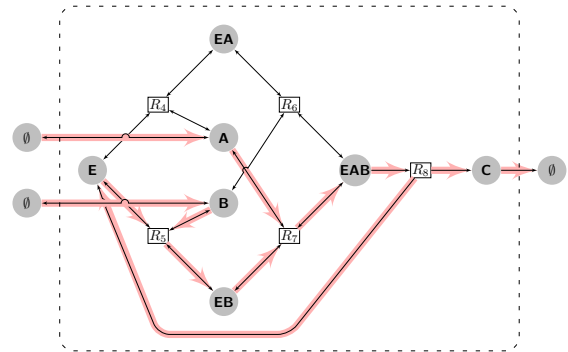
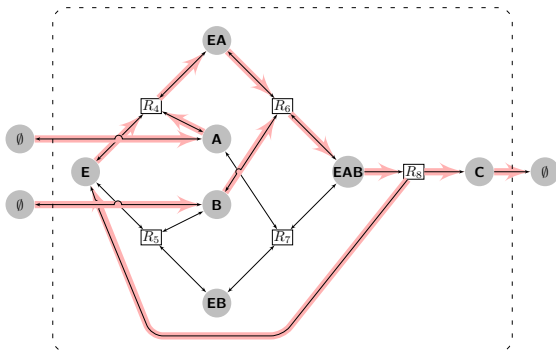
Enzyme Mechanism: Unordered Substrate Binding



$$S = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Engl HW et al (2009), *Inverse Problems* 25:123014 | doi:10.1088/0266-5611/25/12/123014

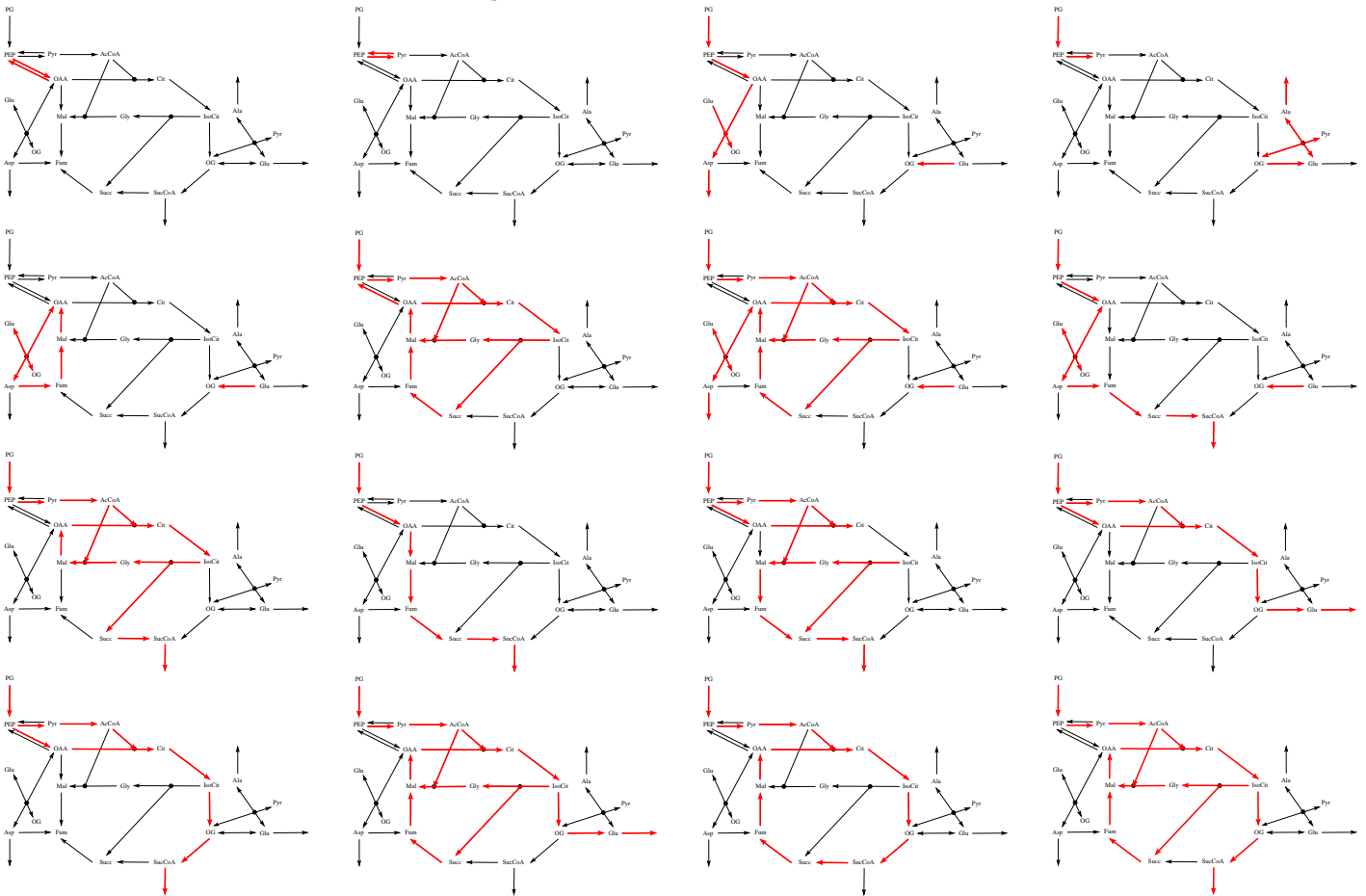
Enzyme Mechanism: Elementary Modes



$$K' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

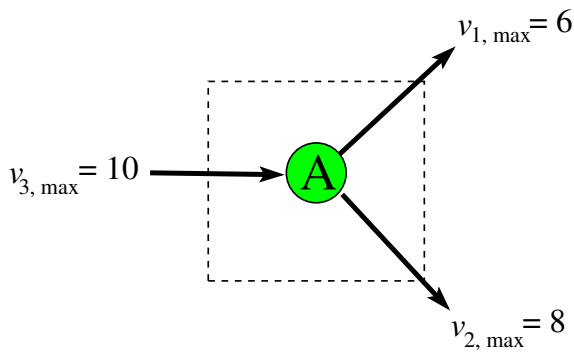
Engl HW et al (2009), *Inverse Problems* 25:123014 | doi:10.1088/0266-5611/25/12/123014

Elementary modes of TCA in *E. coli*

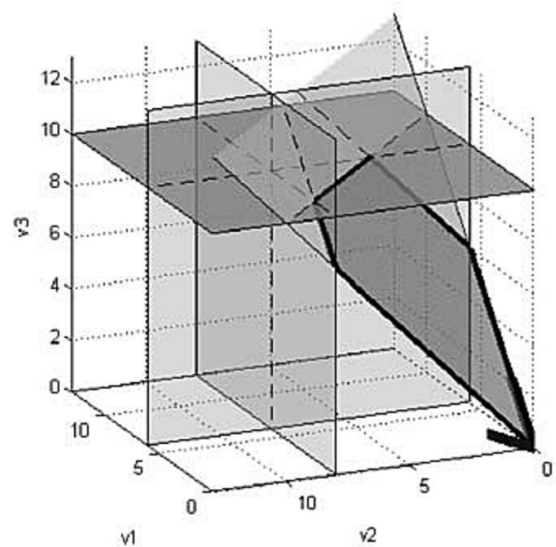


Schuster S et al (1999), Trends Biotechnol 17:53-60 | doi:10.1016/S0167-7799(98)01290-6

Narrowing the Flux Cone: The simple flux split



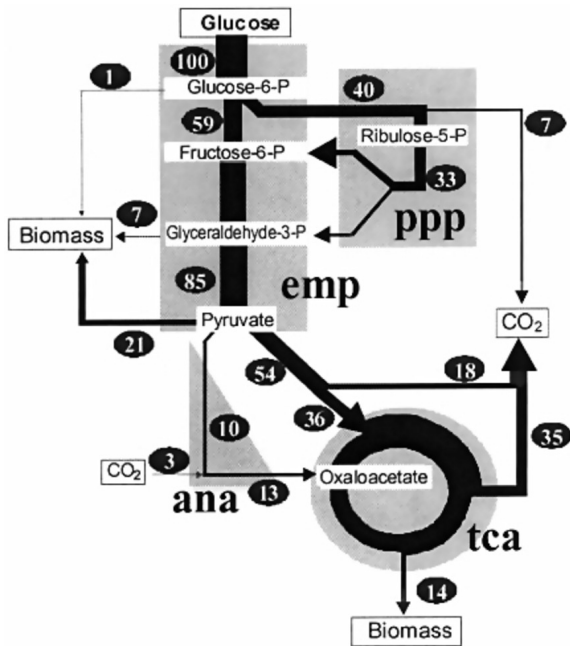
$$S = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$



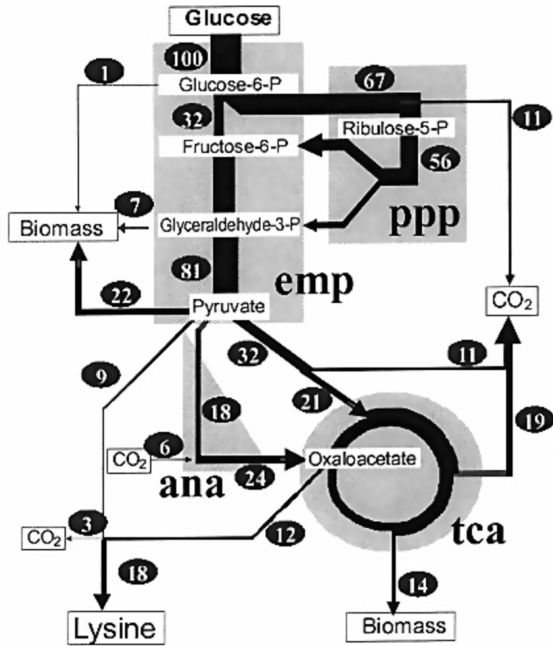
Physico-chemical constraints are required to confine the solution space to functional states the network can achieve.

Price ND et al, (2004) Uniform Sampling of Steady-State Flux Spaces: Means to Design Experiments and to Interpret Enzymopathies, *Biophys J* 87(4):2172-2186 | 10.1529/biophysj.104.043000

Metabolic Flux Analysis (MFA)



normal growth condition.



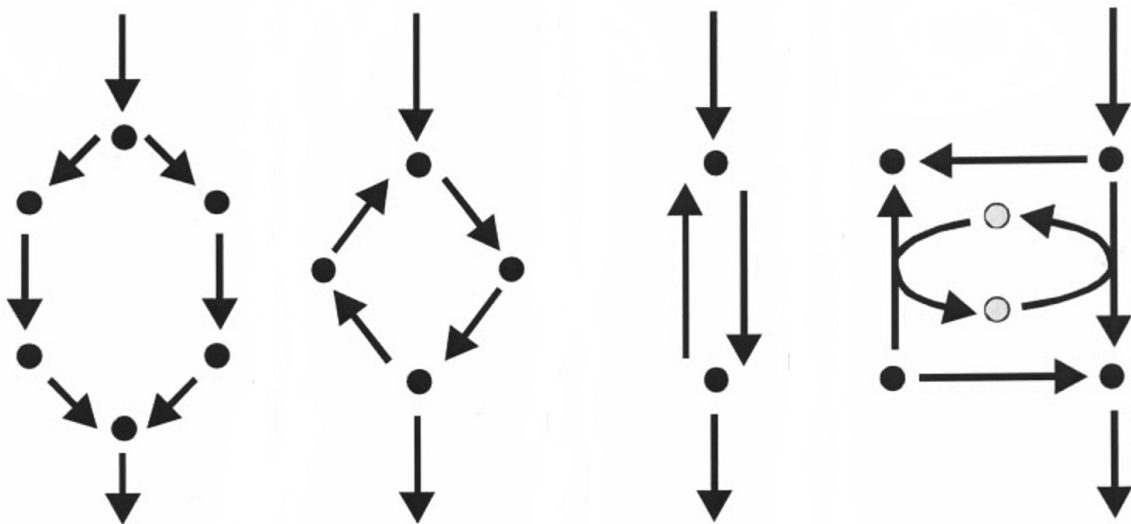
Lys-production condition.

The goal of MFA is the quantitative description of cellular fluxes.

Figure adapted from Marx A, Bestimmung des Kohlenstoffflusses im Zentralstoffwechsel von *Corynebacterium glutamicum* mittels ^{13}C -Isotopenanalyse, PhD-thesis Uni Düsseldorf (1997).

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Fail cases for Metabolic Flux Analysis (MFA)

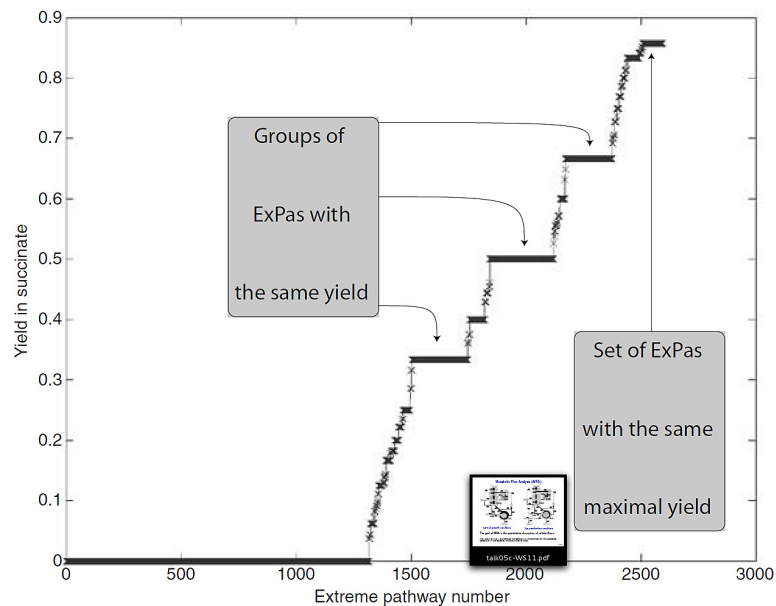


- parallel pathways without any related flux measurement.
- certain metabolic cycles.
- bi-directional reaction steps.
- split pathways when cofactors are **not** balanced.

Figure adapted from [Wiechert, 2001]

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Many Flux Distributions for the same “optimal” yield



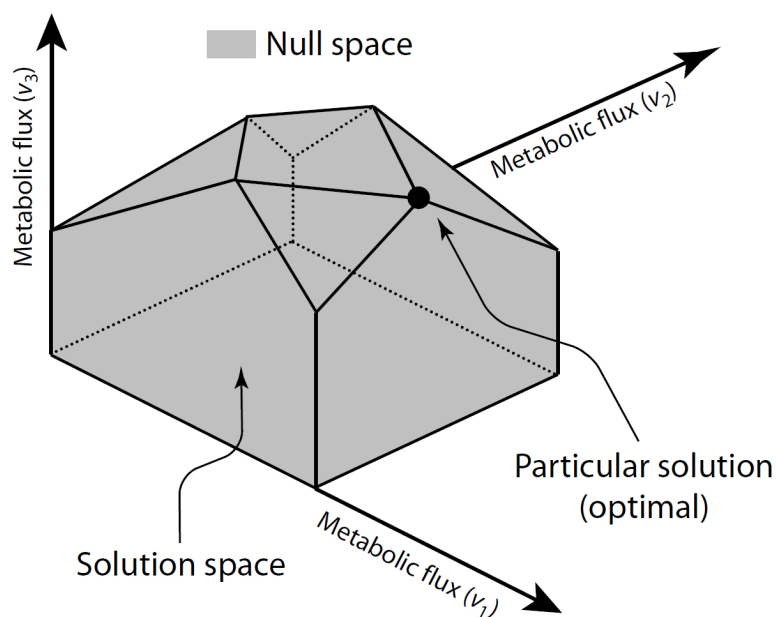
Core metabolic network of *E. coli* (56 compounds, 64 reactions, 2598 extreme pathways).

The optimal value for SUC production (0.86 mol/mol) from FUM can be achieved by 88 different flux distributions!

(Note: any non-negative linear combination is optimal as well)

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Finding the “Best” Flux Distribution



An **objective function** is required to pick out particular solutions with desired properties from the bounded null space.

Figure from Palsson, *BØ Systems Biology, Properties of Reconstructed Networks*, Cambridge University Press, ISBN-13 978-0-521-85903-5

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Flux Balance Analysis

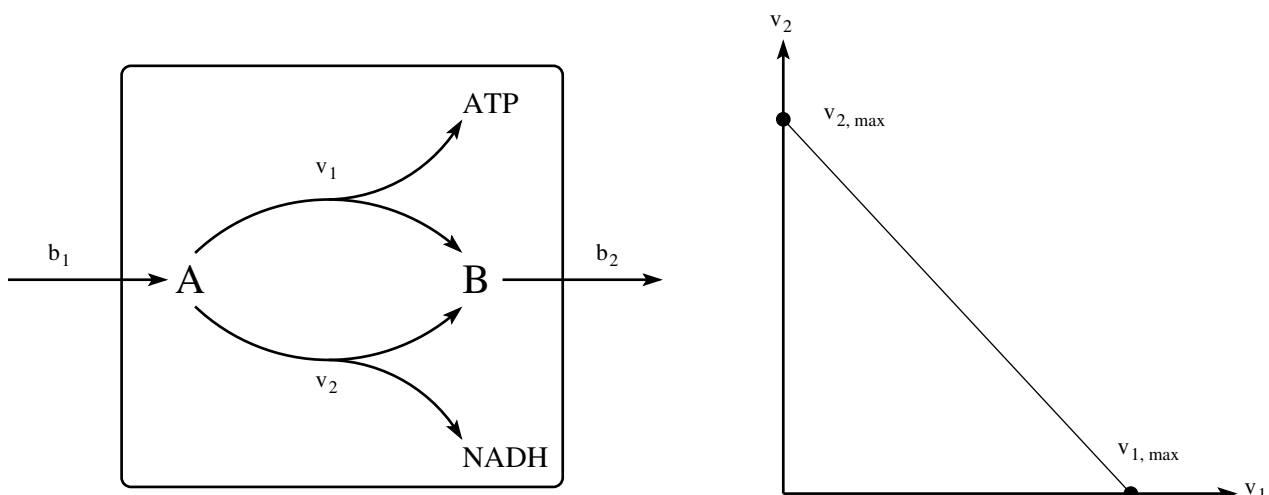
Formulate as an **linear programming problem** with additional constraints (capacity of enzymes, external fluxes, ...) and an appropriate optimization function.

Obj Function	Explanation
$\max \frac{v_{\text{Biomass}}}{v_{\text{Glucose}}}$	biomass yield (same as growth rate)
$\max \frac{v_{\text{ATP}}}{v_{\text{Glucose}}}$	ATP yield
$\min \frac{\sum v_{\text{NADH}}}{v_{\text{Glucose}}}$	redox potential
$\min \sum \delta_i$	reaction steps
$\max \frac{v_{\text{Biomass}}}{\sum v_i^2}$	biomass yield per flux unit

Schuetz R et al (2007), **Systematic evaluation of objective functions for predicting intracellular fluxes in E. coli**, Mol Sys Biol 3:119 | doi:10.1038/msb4100162

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A Simple Example



The flux balance for this system is $v_1 + v_2 = b_1$ ($= b_2$).

$v_1, v_2 \geq 0$ constrains the solution space to a line segment.

Maximizing ATP or NADH production lie at the end of the line.

Example modified from Bonarius HPJ et al (1997), Flux analysis of underdetermined metabolic networks: the quest for the missing constraints, *Trends Biotech* 15(8):308-314 | doi:10.1016/S0167-7799(97)01067-6

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Primer: Linear Programming (LP)

Linear programming is an **optimization** method requiring 2 inputs:

- ① A linear objective function.
- ② A set of linear constraints.

Example: Production planning problem

Product	machine 1	machine 2	machine 3
A	40	24	0
B	24	48	60

Total machine running time is 8 hours/day.

Profit: 10 €/A and 40 €/B.

Question: How many units of product A and B need to be manufactured in order to maximize profit?

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Expressed as LP Problem

- ① maximize profit:

$$z = F(x_1, x_2) = 10 \cdot x_1 + 40 \cdot x_2$$

- ② subject to the linear constraints:

$$40 \cdot x_1 + 24 \cdot x_2 \leq 480$$

$$24 \cdot x_1 + 48 \cdot x_2 \leq 480$$

$$60 \cdot x_2 \leq 480$$

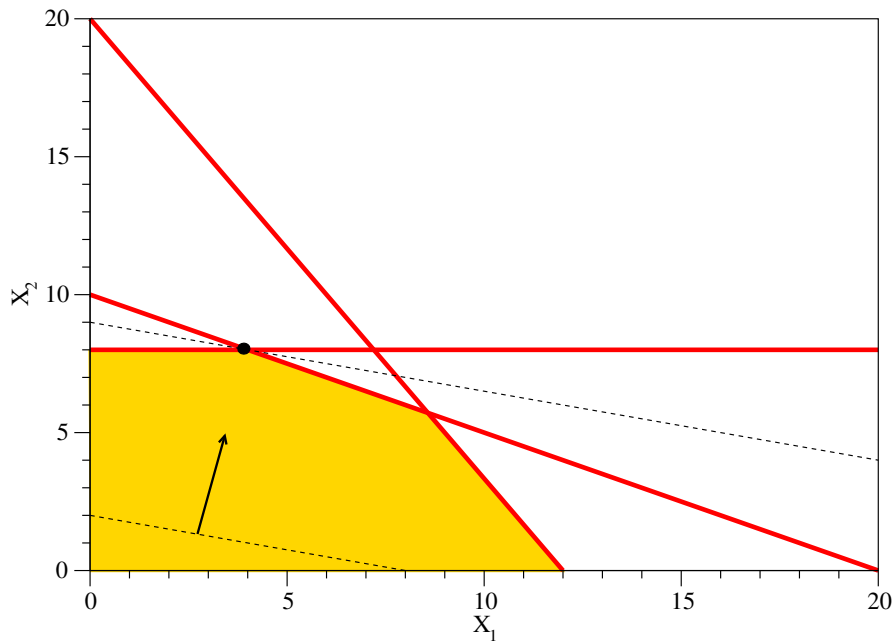
$$x_1, x_2 \geq 0$$

Admissible solutions:

- $x_1 = 0 \wedge x_2 = 0 \implies z = 0$
- $x_2 = 0 \curvearrowright x_1 = 12 \implies z = 120$
- $x_1 = 0 \curvearrowright x_2 = 8 \implies z = 320$

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LP Problem: Graphical Solution



$$\begin{aligned}
 40 \cdot x_1 + 24 \cdot x_2 \leq 480 & \iff x_2 \leq -\frac{5}{3} \cdot x_1 + 20 \\
 24 \cdot x_1 + 48 \cdot x_2 \leq 480 & \iff x_2 \leq -\frac{1}{2} \cdot x_1 + 10 \\
 60 \cdot x_2 \leq 480 & \iff x_2 \leq 8
 \end{aligned}$$

$$z = 10 \cdot x_1 + 40 \cdot x_2 \iff x_2 = -\frac{1}{4} \cdot x_1 + \frac{1}{40} \cdot z$$

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LP Problem: Formal Formulation

The linear objective function is generally a sum of terms that contain weighted measurable elements from a metabolic model.

Maximize:

$$Z = c_1 \cdot x_1 + c_2 \cdot x_2 + \dots = \mathbf{c}^T \cdot \mathbf{x}$$

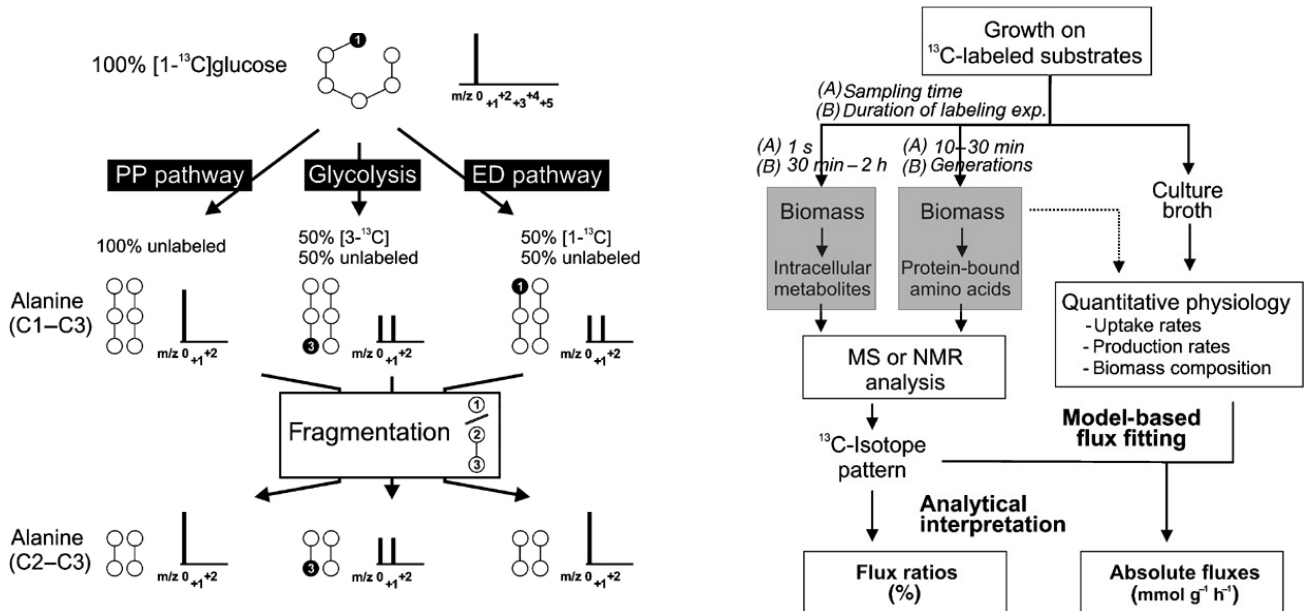
Subject to:

$$\begin{aligned}
 a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots & a_{1n} \cdot x_n \leq b_1 \\
 a_{12} \cdot x_1 + a_{22} \cdot x_2 + \dots & a_{2n} \cdot x_n \leq b_2 \\
 \vdots & \vdots \\
 a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \dots & a_{mn} \cdot x_n \leq b_m
 \end{aligned}$$

$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

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How to measure Fluxes experimentally?

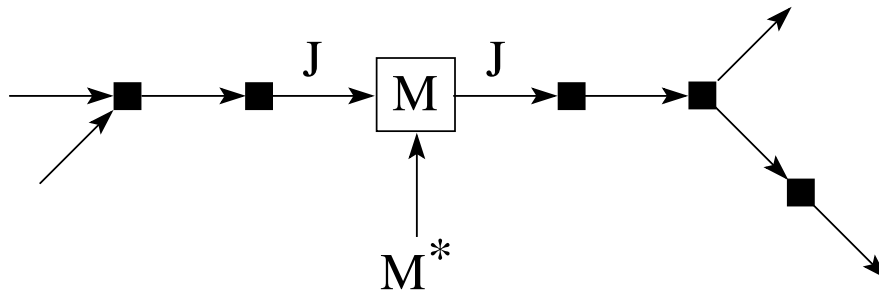


Metabolic fluxes can not be measured directly but must be inferred from **isotopomere patterns**.

Sauer, U (2006), Metabolic network in motion: ¹³C-based flux analysis, *Mol Sys Biol* 2:62 | doi:10.1038/msb4100109

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Direct determination of metabolic flux



$$\frac{dM}{dt} = - \left(\frac{M^*}{M} \right) \cdot J$$

J and M are constant due to metabolic steady-state therefore integration gives

$$\log \left(\frac{M^*(t)}{M^*(0)} \right) = - \frac{J}{M} \cdot t$$

Hence the unknown flux J can be determined from a semilog plot of “radioactive” counts versus time.

Requirements: (1) M^* be transported into the cell (2) intensity of M^* be measurable as a function of time.

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Isotopomers

Are defined as isomers of a metabolite that differ only in the **labeling state** of their individual atoms

(e.g. carbon [^{12}C , ^{13}C], hydrogen [^1H , ^2H] or oxygen [^{16}O , ^{17}O , ^{18}O].)

2^N isotopomers are possible for a metabolite with N atoms that may be in one of two states (unlabeled or labeled).

Example (glucose $\text{C}_6\text{H}_{12}\text{O}_6$)

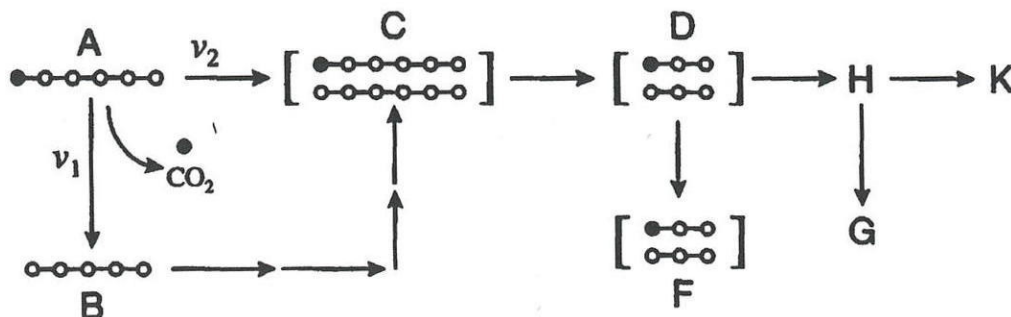
Atoms	# of Isotopomers	
C	6.400×10^1	$(2^6 = 64)$
O	7.260×10^2	$(3^6 = 726)$
H	4.096×10^3	$(2^{12} = 4096)$
C, H	2.621×10^5	$(2^6 \times 2^{12} = 262144)$
C, H, O	1.911×10^8	$(2^6 \times 2^{12} \times 3^6 = 191102976)$

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Determination of Flux Split Ratios

Method works only at both **metabolic** and **isotopic steady state**.

In isotopic steady state the relative population all the isotopomers of a metabolite is constant.

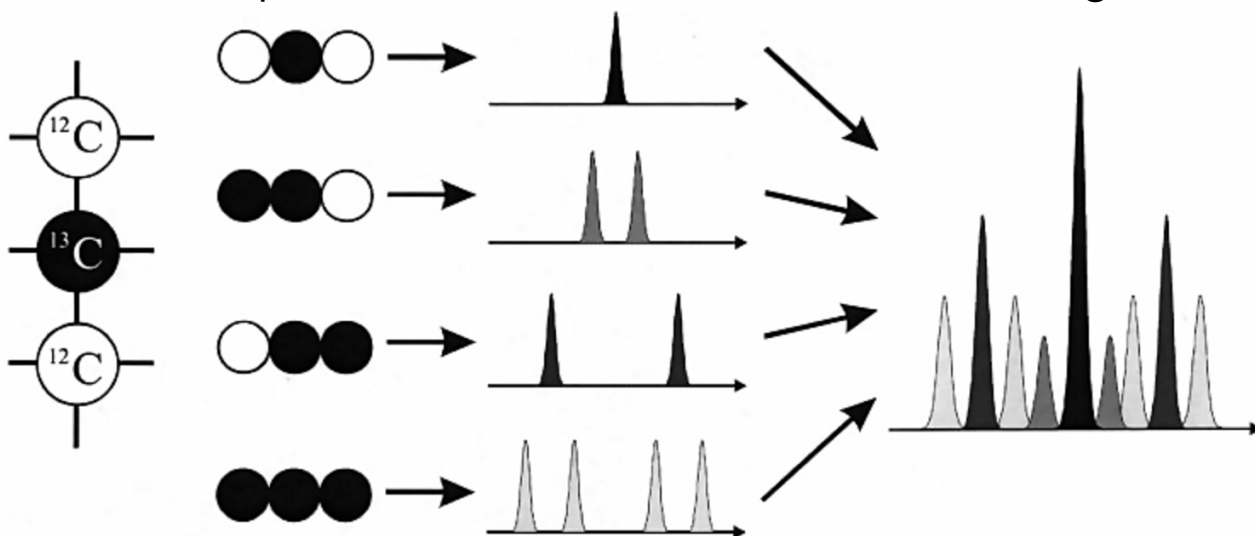


- ① A is a six carbon compound whose first atom is labeled.
- ② Competing/alternative pathways must introduce asymmetries.
- ③ The pathway via intermediate B produces unlabeled C.
- ④ Via the “direct” pathway the label is retained.
- ⑤ The label enrichment in C is directly proportional to the rate of v_2 relative to the total rate of A consumption ($v_1 + v_2$).

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Measuring Isotopomere Distribution

Any experimental technique capable of detecting differences between isotopomers can be used to measure the labeling state.



The two dominating technologies are:

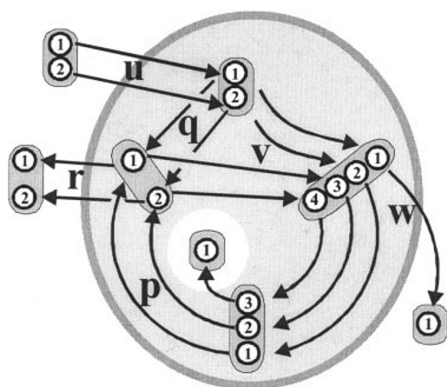
- 1 Nuclear magnetic resonance spectroscopy (NMR).
- 2 Mass spectrometry (MS).

Figure adapted from [Wiechert, 2001]

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Atom Transition Network

Estimating fluxes from isotopomere patterns is an **inverse problem**.

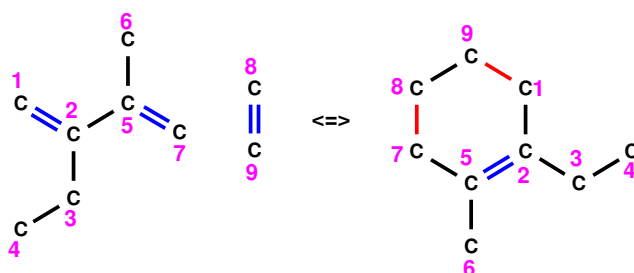


The **atom-atom mapping** between reaction educts and products must be known.

Getting this information is an **NP-hard problem**.

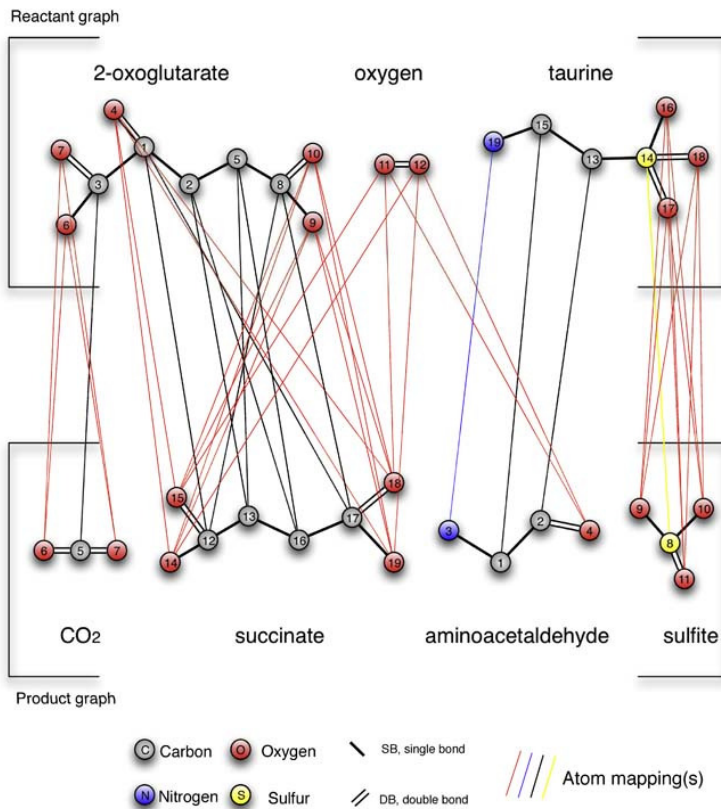
Therefore most approaches use a heuristic optimization principle e.g.

“minimal chemical distance”.



Degeneracy of Atom-Atom Mappings

Taurine dioxygenase [c]: akg + o2 + taur --> aacald + co2 + h + so3 + succ



Increasing degeneracy:

- Rotational symmetric molecules.
- Equivalence of oxygen atoms in carboxyl or phosphate groups.

Restricting degeneracy:

- Prochiral carbone centers.
- Molecules with a center of inversion but lacking of a rotational axis.

Ravikirthi P et al (2011), Construction of an *E. Coli* Genome-Scale Atom Mapping Model for MFA Calculations, *Biotechnol Bioeng* **108**:1372-1382 | doi:10.1002/bit.23070

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Further Reading



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